Find the inverse of the following matrices by partitioning:

4. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{10}\begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ **Ans.** $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 4 & 2 & 7 \\ 2 & 3 & 3 & 2 \\ 52 & 7 & 3 & 9 \\ 2 & 3 & 2 & 3 \end{bmatrix}$ **Ans.** $\frac{1}{2}\begin{bmatrix} -1 & 11 & 7 & -26 \\ -1 & -7 & -3 & 16 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 2 \end{bmatrix}$

4.25 CHARACTERISTIC ROOTS OR EIGEN VALUES

(*a*) For a given square matrix A, $A - \lambda I$ matrix is called the *characteristic matrix*, where λ is scalar and I is the unit matrix.

Let
$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

characteristic matrix

(*b*) **Characteristic Polynomial**

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

For example
$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$
$$= (2-\lambda)(6 - 5\lambda + \lambda^2 - 2) - 2(2 - \lambda - 1) + 1(2 - 3 + \lambda)$$
$$= \lambda^3 - 7\lambda^2 + 11\lambda - 5$$

(*c*) **Characteristic Equation**

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A *e.g.*
$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

(*d*) **Characteristic Roots or Eigenvalues**

The roots of characteristic equation $|A - \lambda I| = 0$ are called characteristic roots of matrix A. *e.g.*
$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$
$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0 \quad \therefore \quad \lambda = 1, 1, 5$$

Characteristic roots are 1, 1, 5.

Some Important Properties of Eigenvalues

(1) Any square matrix A and its transpose A' have the same eigenvalues.

Note. The sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.

(2) The sum of the eigenvalues of a matrix is equal to the trace of the matrix.

(3) The product of the eigenvalues of a matrix A is equal to the determinant of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A, then the eigenvalues of

(i) kA are $k\lambda_1, k\lambda_2, \ldots\ldots, k\lambda_n$ (ii) A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots, \lambda_n^m$

(iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots, \frac{1}{\lambda_n}$. (A.M.I.E.T.E., Winter 1997)

Example 59. *Find the characteristic roots of the matrix*

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution. The characteristic equation of the given matrix is

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

or $$(6-\lambda)(9-6\lambda+\lambda^2-1)+2(-6+2\lambda+2)+2(2-6+2\lambda) = 0$$

or $$-\lambda^3+12\lambda^2-36\lambda+32 = 0$$

By trial, $\lambda = 2$ is a root of this equation.

or $$(\lambda-2)(\lambda^2-10\lambda+16) = 0 \quad \text{or} \quad (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

or $\lambda = 2, 2, 8$ are the characteristic roots or Eigen values. **Ans.**

Example 60. *The matrix A is defined as*

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Find the eigenvalues of $3A^3+5A^2-6A+2I$.

Solution. $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} = 0$$

or $$(1-\lambda)(3-\lambda)(-2-\lambda) = 0 \quad \text{or} \quad \lambda = 1, 3, -2$$

Eigenvalues of $A^3 = 1, 27, -8$; Eigenvalues of $A^2 = 1, 9, 4$

Eigenvalues of $A = 1, 3, -2$; Eigenvalues of $I = 1, 1, 1$

$\therefore$ Eigenvalues of $3A^3+5A^2-6A+2I$

First eigenvalue $= 3(1)^2+5(1)^2-6(1)+2 = 4$

Second eigenvalue $= 3(27)+5(9)-6(3)+2(1) = 110$

Third eigenvalue $= 3(-8)+5(4)-6(-2)+2(1) = 10$

Required eigenvalues are 4, 110, 10. **Ans.**

Example 61. *If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$.*

Solution. $(A - \lambda I)^2 = A^2 - 2\lambda AI + \lambda^2 I^2 = A^2 - 2\lambda A + \lambda^2 I$

Eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^2 \ldots \lambda_n^2$

Eigenvalues of $2\lambda A$ are $2\lambda\lambda_1, 2\lambda\lambda_2, 2\lambda\lambda_3 \ldots 2\lambda\lambda_n$.

Eigenvalues of $\lambda^2 I$ are λ^2,

$\therefore$ Eigenvalues of $A^2 - 2\lambda A + \lambda^2 I$ are

$$\lambda_1^2 - 2\lambda\lambda_1 + \lambda^2, \lambda_2^2 - 2\lambda\lambda_2 + \lambda^2, \lambda_3^2 - 2\lambda\lambda_3 + \lambda^2 \ldots$$

or $$(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, (\lambda_3 - \lambda)^2, \ldots (\lambda_n - \lambda)^2$$ **Ans.**

Example 62. *Prove that the following matrices have the same characteristic equation.*

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

Solution. Characteristic equation of first matrix is $|A - \lambda I| = 0$

or $$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0.$$

$$(a-\lambda)[\lambda^2 - \lambda(b+c) + bc - a^2] - b(b^2 - ac - b\lambda) + c(ab - c^2 + c\lambda) = 0$$

or $$-\lambda^3 + \lambda^2(a+b+c) + \lambda(-ab - ac - bc + a^2 + b^2 + c^2) + (abc - a^3 - b^3 + abc + abc - c^3) = 0$$

or $$\lambda^3 - \lambda^2(a+b+c) - \lambda(a^2 + b^2 + c^2 - ab - bc - ca) + (a^3 + b^3 + c^3 - 3abc) = 0.$$

The symmetry of the result shows that characteristic equation for the other two matrices will also be same.

Proved

Example 63. *Prove that a matrix A and its transpose A′ have the same characteristic roots.*

Solution. Characteristic equation of matrix A is

$$|A - \lambda I| = 0. \quad \ldots(1)$$

Characteristic equation of matrix A' is

$$|A' - \lambda I| = 0 \quad \ldots(2)$$

Clearly both (1) and (2) are same, as we know that

$$|A| = |A'|$$

i.e., a determinant remains unchanged when rows be changed into columns and columns into rows.

Proved

Example 64. *If A and P be square matrices of the same type and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic roots.*

(A.M.I.E.T.E., *Summer 1998, Winter 1996*)

Solution. Let us put $B = P^{-1}AP$ and we will show that characteristic equations for both A and B are the same and hence they have the same characteristic roots.

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$$

$$\therefore \quad |B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P|$$
$$= |A - \lambda I||P^{-1}||P| = |A - \lambda I||P^{-1}P|$$
$$= |A - \lambda I||I| = |A - \lambda I| \text{ as } |I| = 1.$$

Thus the matrices A and B have the same characteristic equations and hence same characteristic roots. **Proved**

Example 65. *If A and B be two square invertible matrices, then prove that AB and BA have the same characteristic roots.*

Solution. Now $\quad AB = IAB = B^{-1}B(AB) = B^{-1}(BA)B \quad$ ...(1)

But by Ex. 64, matrices BA and $B^{-1}(BA)B$ have same characteristic roots or matrices BA and AB by (1) have same characteristic roots. **Proved**

Example 66. *If A and B be n rowed square matrices and if A be invertible, show that the matrices $A^{-1}B$ and BA^{-1} have the same characteristic roots.*

Solution. $\quad A^{-1}B = A^{-1}BI = A^{-1}B(A^{-1}A) = A^{-1}(BA^{-1})A. \quad$ ...(1)

But by Ex. 64, matrices BA^{-1} and $A^{-1}(BA^{-1})A$ have same characteristic roots or matrices BA^{-1} and $A^{-1}B$ by (1) have same characteristic roots. **Proved**

Example 67. *Show that 0 is a characteristic root of a matrix, if and only if, the matrix is singular.*

Solution. Characteristic equation of matrix A is given by

$$|A - \lambda I| = 0.$$

If $\lambda = 0$ then from above it follows that $|A| = 0$ *i.e.* Matrix A is singular.

Again if Matrix A is singular *i.e.*, $|A| = 0$ then

$$|A - \lambda I| = 0 \Rightarrow |A| - \lambda|I| = 0, 0 - \lambda \cdot 1 = 0 \Rightarrow \lambda = 0.$$ **Proved**

Example 68. *Show that characteristic roots of a triangular matrix are just the diagonal elements of the matrix.*

Solution. Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Characteristic equation is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} a_{11} - \lambda & 0 & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 & 0 \\ a_{31} & a_{32} & a_{33} - \lambda & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} - \lambda \end{vmatrix} = 0$$

On expansion it gives

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)(a_{44} - \lambda) = 0$$

$$\therefore \quad \lambda = a_{11}, a_{22}, a_{33}, a_{44}$$

which are diagonal elements of matrix A. **Proved**

Example 69. *The characteristic roots of a Hermitian matrix are all real.*

Solution. We know that matrix A is Hermitian if

$$A^{\theta} = A \text{ i.e., where } A^{\theta} = (\overline{A'}) \text{ or } (\overline{A})'.$$

Also

$$(\lambda A)^\theta = \bar{\lambda} A^\theta \text{ and } (AB)^\theta = B^\theta A^\theta.$$

If λ is a characteristic root of matrix A then $AX = \lambda X$. ...(1)

$\therefore$
$$(AX)^\theta = (\lambda X)^\theta \quad \text{or} \quad X^\theta A^\theta = \lambda X^\theta.$$

But A is Hermitian $\therefore$ $A^\theta = A$.

$\therefore$
$$X^\theta A = \bar{\lambda} X^\theta \quad \therefore \quad X^\theta AX = \bar{\lambda} X^\theta X. \qquad ...(2)$$

Again from (1)
$$IX^\theta AX = X^\theta \lambda X = \lambda X^\theta X. \qquad ...(3)$$

Hence from (2) and (3) we conclude that $\bar{\lambda} = \lambda$ showing that λ is real.

Deduction 1. *From above we conclude that characteristic roots of real symmetric matrix are all real, as in this case, real symmetric matrix will be Hermitian.*

For symmetric, we know that $A' = A$. $\quad (\bar{A}') = \bar{A}$.

or
$$A^\theta = A \quad \therefore \quad \bar{A} = A \text{ as } A \text{ is real. Rest as above.}$$

Deduction 2. *Characteristic of a skew Hermitian matrix is either zero or a pure imaginary number.*

If A is skew Hermitian then iA is Hermitian.

Also λ be a characteristic root of A then $AX = \lambda X$.

$\therefore$
$$(i \cdot A) X = (i \lambda) X.$$

Above shows that $i\lambda$ is characteristic root of matrix iA, which is Hermitian and hence $i\lambda$ should be real, which will be possible if λ is either pure imaginary or zero.

Example 70. *The modulus of each characteristic root of a unitary matrix is unity.*

Solution. If A be a unitary matrix, then we know $A^\theta A = I$.

If λ be characteristic root of A, then $AX = \lambda X$. ...(1)

$$(AX)^\theta = (\lambda X)^\theta \text{ or } X^\theta A^\theta = \bar{\lambda} X^\theta. \qquad ...(2)$$

$$X^\theta A^\theta AX = \lambda X^\theta \lambda X = \bar{\lambda} X \lambda X^\theta X.$$

But
$$A^\theta A = I.$$

$$X^\theta IX = \bar{\lambda}\lambda X^\theta X \text{ or } X^\theta X = \bar{\lambda}\lambda X^\theta X$$

$\therefore$
$$(1 - \lambda\bar{\lambda}) X^\theta X = 0.$$

But $X \neq 0 \therefore X^\theta X \neq 0$ and hence $1 - \lambda\bar{\lambda} = 0$ or $1 = \bar{\lambda} \cdot \lambda$ or $|\lambda|^2 = 1$.

i.e. characteristic roots are unimodular.

Example 71. *If λ is an eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is also eigenvalue.*

(A.M.I.E.T.E., *Winter 1995*)

[**Hint:** $AA' = I$ if λ is the eigen value of A, then $\lambda^2 = I$, $\lambda = \frac{1}{\lambda}$]

4.26 CAYLEY-HAMILTON THEOREM

Statement. Every square matrix satisfies its own characteristic equation.

If $|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n)$ be the characteristic polynomial of $n \times n$ matrix $A = (a_{ij})$, then the matrix equation

$$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots + a_n I = 0 \text{ is satisfied by } X = A \text{ i.e.,}$$

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$$

Proof. Since the elements of $A - \lambda I$ are at most of the first degree in λ, the elements of adj $(A - \lambda I)$ are at most degree $(n-1)$ in λ. Thus, *adj* $(A - \lambda I)$ may be written as a matrix polynomial in λ, given by λ,

$$Adj\,(A - \lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \ldots\ldots + B_{n-1}$$

where $B_0, B_1 \ldots . B_{n-1}$ are $n \times n$ matrices, their elements being polynomial in λ.

We know that

$$(A - \lambda I)\, Adj\,(A - \lambda I) = |A - \lambda I|\, I$$

$$(A - \lambda I)(B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \ldots + B_{n-1}) = (-1)^n\,(\lambda^n + a_1 \lambda^{n-1} + \ldots + a_n)\, I$$

Equating coefficients of like powers of λ on both sides we get

$$-IB_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n a_2 I$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$AB_{n-1} = (-1)^n a_n I$$

On multiplying the equation by $A^n, A^{n-1}, \ldots, I$ respectively and adding, we obtain

$$0 = (-1)^n\,[A^n + a_1 A^{n-1} + \ldots . + a_n I]$$

Thus $\quad A^n + a_1 A^{n-1} + \ldots . + a_n I = 0$ **Proved**

Thus if A is square matrix and if

$$\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0 \qquad \ldots (1)$$

be its characteristic equation, then according to Cayley Hamilton Theorem (1) is satisfied by A.

$$A^3 - 2A^2 + 3A - 4I = 0 \qquad \ldots (2)$$

We can find out A^{-1} from (2). On premultiplying (2) by A^{-1} we get

$$A^2 - 2A + 3I - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}[A^2 - 2A + 3I]$$

Example 72. *Find the characteristic equation of the matrix*

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Hence find A^{-1}. (*A.M.I.E.T.E., Summer 1996*)

Solution. Characteristic equation of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

or $\quad (1-\lambda)[(1-\lambda)(-1-\lambda) - 3] - 2(-1-\lambda-1) - 2(3-1+\lambda) = 0$

or $\quad (1-\lambda)(-1+\lambda^2-3) - 2(-\lambda-2) - 2(2+\lambda) = 0$

or $\quad (1-\lambda)(\lambda^2-4) + 2\lambda + 4 - 4 - 2\lambda = 0$

or $\quad (-\lambda+1)(\lambda^2-4) = 0 \quad$ or $\quad \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

By Cayley-Hamilton Theorem

or $$A^3 - A^2 - 4A + 4I = 0$$

or $$A^2 - A - 4I + 4A^{-1} = 0 \qquad \text{(Multiplying by } A^{-1}\text{)}$$

$$4A^{-1} = [-A^2 + A + 4I] \qquad ...(1)$$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

From (1), we have

$$4A^{-1} = -\begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1+4 & 2+2+0 & -2-2+0 \\ -3+1+0 & -6+1+4 & 2+1+0 \\ -3+1+0 & -2+3+0 & -2-1+4 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \qquad \textbf{Ans.}$$

Example 73. *Find the characteristic equation of the symmetric matrix*

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by A and hence obtain A^{-1}.

Express $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$ in linear polynomial in A.

(A.M.I.E.T.E., Summer 2000)

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)^2 - 1] + 1[-2+\lambda+1] + 1[1-2+\lambda] = 0$$

or $$(2-\lambda)^3 - (2-\lambda) + \lambda - 1 + \lambda - 1 = 0$$

or $$(2-\lambda)^3 - 2 + \lambda + \lambda - 1 + \lambda - 1 = 0 \text{ or } (2-\lambda)^3 + 3\lambda - 4 = 0$$

or $$8 - \lambda^3 - 12\lambda + 6\lambda^2 + 3\lambda - 4 = 0$$

or $$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0 \text{ or } \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

By Cayley-Hamilton Theorem $A^3 - 6A^2 + 9A - 4I = 0$ ...(1)

Verification :

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{pmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$A^3 - 6A^2 + 9A - 4I$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So it is verified that the characteristic equation (1) is satisfied by A.

Inverse of Matrix A,

$$A^3 - 6A^2 + 9A - 4I = 0$$

On multiplying by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0 \quad \text{or} \quad 4A^{-1} = A^2 - 6A + 9I$$

or $$4A^{-1} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix}, \quad A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$ **Ans.**

$A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$

$= A^3(A^3 - 6A^2 + 9A - 4I) + 2(A^3 - 6A^2 + 9A - 4I) + 5A - I$

$= 5A - I$ **Ans.**

Example 74. *Find the characteristic equation of the matrix A*

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

Hence find A^{-1}.

Solution. Characteristic equation is

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[1+\lambda^2-2\lambda+4] - 3(2-2\lambda+2) + 1\cdot(4-1+\lambda) = 0$$

or $$(4-\lambda)(\lambda^2-2\lambda+5) - 3(-2\lambda+4) + (3+\lambda) = 0$$

or $$4\lambda^2 - 8\lambda + 20 - \lambda^3 + 2\lambda^2 - 5\lambda + 6\lambda - 12 + 3 + \lambda = 0$$

or

$$-\lambda^3+6\lambda^2-6\lambda+11=0 \quad \text{or} \quad \lambda^3-6\lambda^2+6\lambda-11=0$$

By Cayley Hamilton Theorem

$$A^3-6A^2+6A-11I=0$$

By multiplying by A^{-1} we get

$$A^2-6A+6I-11A^{-1}=0 \quad \text{or} \quad 11A^{-1}=A^2-6A+6I$$

$$11A^{-1}=\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}-6\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}+6\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}23&17&-1\\8&3&-2\\9&7&-2\end{bmatrix}+\begin{bmatrix}-24&-18&-6\\-12&-6&12\\-6&-12&-6\end{bmatrix}+\begin{bmatrix}6&0&0\\0&6&0\\0&0&6\end{bmatrix}$$

$$=\begin{bmatrix}5&-1&-7\\-4&3&10\\3&-5&-2\end{bmatrix} \quad \therefore \ A^{-1}=\frac{1}{11}\begin{bmatrix}5&-1&-7\\-4&3&10\\3&-5&-2\end{bmatrix}$$ **Ans.**

Example 75. *Find the characteristic equation of the matrix*

$$A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$$

Verify Cayley-Hamilton theorem and hence evaluate the matrix equation.

$$A^8-5A^7+7A^6-3A^5+A^4=5A^3-8A^2+2A-I.$$ (*A.M.I.E.T.E.*, **Winter 2001**, *1999*)

Solution. Characteristic equation of the matrix A is

$$\begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0 \text{ or } (2-\lambda)[(1-\lambda)(2-\lambda)]-1(0)+1(0-1+\lambda)=0$$

or

$$\lambda^3-5\lambda^2+7\lambda-3=0$$

According to Cayley-Hamilton Theorem

$$A^3-5A^2+7A-3I=0 \qquad \dots(1)$$

We have to verify the eq. (1).

$$A^2=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$$

$$A^3=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}=\begin{bmatrix}14&13&13\\0&1&0\\13&13&14\end{bmatrix}$$

$$A^3-5A^2+7A-3I=\begin{bmatrix}14&13&13\\0&1&0\\13&13&14\end{bmatrix}-5\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+7\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}-3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}14-25+14-3&13-20+7+0&13-20+7+0\\0+0+0+0&1-5+7-3&0-0+0-0\\13-20+7+0&13-20+7-0&14-25+14-3\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

Hence, Cayley Hamilton Theorem is verified.

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= A^5 XO + AXO + A^2 + A + I = + A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$ **Ans.**

Exercise 4.11

1. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Verify Cayley Hamilton Theorem for this matrix. Hence find A^{-1}.

Ans. $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$

2. Use Cayley Hamilton Theorem to find the inverse of the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Ans. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. Using the Cayley-Hamilton Theorem, find A^{-1}, given that

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

(*A.M.I.E.T.E., Summer 1998*) **Ans.** $-\frac{1}{5}\begin{bmatrix} 4 & -5 & -2 \\ 7 & -10 & -1 \\ -2 & 0 & 1 \end{bmatrix}$

4. Using Cayley Hamilton Theorem, find the inverse of the matrix

$$\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$$

Ans. $\frac{1}{10}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

5. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

and show that the equation is also satisfied by A. **Ans.** $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

6. Find the eigenvalues of the matrix

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

(*A.M.I.E., Summer 1995*) **Ans.** Eigenvalues are $0, +1, -2$.

7. Using, Cayley-Hamilton relation obtain the inverse of the matrix (*A.M.I.E., Summer 2002*)

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Ans. $\frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$

8. Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

satisfies its characteristic equation. Hence find A^{-1}.

(*A.M.I.E.T.E., Winter 1995*) **Ans.** $\frac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

9. Use Cayley Hamilton Theorem to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

Ans. $A^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$

10. Verify Cayley-Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

Hence evaluate A^{-1}.

Ans. $\frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$

11. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

(*A.M.I.E.T.E., Winter 2001*) **Ans.** $A + 5I$

12. If λ_1, λ_2 and λ_3 are the eigenvalues of the matrix

$$\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$$ then $\lambda_1 + \lambda_2 + \lambda_3$ is equal to

(*i*) −16 (*ii*) 2 (*iii*) −6 (*iv*) −14 (*A.M.I.E.T.E., Winter 1997*) **Ans.** (*ii*)

13. The matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ is given. The eigenvalues of $4A^{-1} + 3A + 2I$ are

(A) 6, 15; (B) 9, 12 (C) 9, 15; (D) 7, 15 (*A.M.I.E.T.E., Winter 1996*) **Ans.** (C)

14. The matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

Find the eigenvalues of $3A^3 + 5A^2 + 6A + I$.

Ans. 15, −15, −53

4.27 CHARACTERISTIC VECTORS OR EIGENVECTORS

Let A be a $n \times n$ square matrix and Y and X are two non-zero column vectors such that

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ x_2 \\ \vdots \\ X_n \end{bmatrix}$$

$Y = AX \Rightarrow A$ transformsvector X to vector Y.

Two vectors X and Y have the same direction. Here we have to determine those vectors X whose images Y are given by

$$Y = \lambda X.$$

Corresponding to each characteristic root λ we have a corresponding non-zero vector X which satisfies the equation $|A - \lambda I| X = 0$. The non-zero vector X is called characteristic vector or Eigenvectors.

19.28 PROPERTIES OF EIGENVECTORS

1. The eigenvector X of a matrix A is not unique.

2. If $\lambda_1, \lambda_2, \ldots \lambda_n$ be distinct eigenvalues of an $n \times n$ matrix then corresponding eigen = vectors $X_1, X_2, \ldots X_n$ form a linearly independent set.

3. If two or more eigenvalues are equal it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.

4. Two eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1' X_2 = 0$.

5. Eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.

Normalised form of vectors. To find normalised form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we divide each element by $\sqrt{a^2+b^2+c^2}$.

For example, normalised form of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

4.29 NON SYMMETRIC MATRICES WITH NON REPEATED EIGEN VALUES

Example 76. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Solution.

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0 \quad i.e., \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Let $\lambda = 1$, $\quad 1 - 6 + 11 - 6 = 0$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -6 & +11 & -6 \\ & & +1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 1, 2, 3$$

To find eigenvectors for the corresponding eigenvalues we will consider the matrix equation

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots (1)$$

Eigen vector corresponding to eigenvalue $\lambda = 1$

By putting $\lambda = 1$, the matrix equation (1) will become

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left\{ \begin{aligned} -z &= 0 \\ x+y+z &= 0 \\ k+y+0 &= 0 \end{aligned} \right\} \quad \text{Let } x = k \;\Rightarrow y = -k$$

Eigen vector X_1 is $\begin{bmatrix} k \\ -k \\ 0 \end{bmatrix}$ or $k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to eigenvalue $\lambda = 2$.

On putting $\lambda = 2$ in eq. (1), it will become

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -x + 0y - z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}$$

$$\frac{x}{0+2} = \frac{y}{-2+1} = \frac{z}{-2-0} \text{ or } \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$$

Eigenvector $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

Eigenvector corresponding to eigenvalue $\lambda = 3$.

On putting $\lambda = 3$ in (1), the equation will become

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -2x + 0y - z &= 0 \\ x - y + z &= 0 \end{aligned} \Rightarrow \frac{x}{0-1} = \frac{y}{-1+2} + \frac{z}{2-0}$$

or

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{-2}$$

Eigenvector $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. **Ans.**

Exercise 4.12

Non-symmetric matrix with different eigen values

Find the eigenvalues and the corresponding eigen vectors for the following matrices

1. $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ **Ans.** $-1, 1, 2$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ **Ans.** $1, 2, 5$; $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ **Ans.** $-2, 1, 3$; $\begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$ **Ans.** $-1, 1, 2$; $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ *(A.M.I.E.T.E., Winter 1996)* **Ans.** $-1, 1, 4$; $\begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ **Ans.** $0, 1, 5$; $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$

7. $\begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$ **Ans.** $-1, 1, 2;$ $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

4.30 NON-SYMMETRIC MATRIX WITH REPEATED EIGEN VALUES

Example 77. *Find all the Eigenvalues and Eigenvectors of the matrix*

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$ *(A.M.I.E.T.E., Winter 1996)*

Solution. Characteristic equation of A is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

or $$(-2-\lambda)[-\lambda^2+\lambda-12]-2(-2\lambda-6)-3(-4+1-\lambda) = 0$$

or $$\lambda^3+\lambda^2-21\lambda-45 = 0 \qquad \text{... (1)}$$

If $\lambda = -3$, $-27+9+63-45 = 0$, so $(\lambda+3)$ is one factor of (1). The remaining factors are obtained on dividing (1) by $\lambda+3$.

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$$\lambda^2-2\lambda-15 = 0 \text{ or } (\lambda-5)(\lambda+3) = 0$$

$$(\lambda+3)(\lambda+3)(\lambda-5) = 0 \qquad \Rightarrow \lambda = 5, -3, -3$$

To find the eigenvectors for corresponding eigenvalues, we will consider the matrix equation

$$(A-\lambda I)X = 0 \text{ i.e., } \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{... (2)}$$

On putting $\lambda = 5$ in eq. (2), it becomes

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $-7x+2y-3z = 0$, $2x-4y-6z = 0$, $-x-2y-5z = 0$

From first and second equations, we have

$$\frac{x}{-12-12} = \frac{y}{-6-42} = \frac{z}{28-4} \text{ or } \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = k$$

Hence the eigenvector is $X_1 = \begin{bmatrix} k \\ 2k \\ -k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Put $\lambda = -3$ in eq. (2)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $x+2y-3z = 0$, $2x+4y-6z = 0$, $-x-2y+3z = 0$

First, second and third equations are the same.

Let $x = k_1,\ y = k_2$ then $z = \frac{1}{3}(k_1 + 2k_2)$

Hence, the eigenvector is $\begin{bmatrix} k_1 \\ k_2 \\ \frac{1}{3}(k_1 + 2k_2) \end{bmatrix}$

Let $k_1 = 0,\ k_2 = 3$, Hence $X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

Since the matrix is non-symmetric the corresponding eigenvectors X_2 and X_3 must be linearly independent. This can be done by choosing

$k_2 = 0$, and $k_1 = 1$, Hence $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_1 = 5 \to X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\ \lambda = -3 \to X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix},\ \lambda = -3 \to X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. **Ans.**

Example 78. *Show that the matrix* $A \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ *has less than three linearly independent eigenvectors. Is it possible to obtain a similarity transformation that will diagonalise this matrix?*

(A.M.I.E.T.E., Summer 1996)

Solution. $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

Let $\lambda = 2,\quad 8 - 28 + 32 - 12 = 0.\ \therefore (\lambda - 2)$ is one factor.

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda^2 - 5 + 6 = 0$$

$$(\lambda - 2)(\lambda^2 - 5\lambda + 6) = 0 \text{ or } (\lambda - 2)(\lambda - 2)(\lambda - 3) = 0,\ \lambda = 2, 2, 3$$

Eigenvector for $\lambda = 3$

$$\begin{bmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0x + 10y + 5z &= 0 \\ -2x - 6y - 4z &= 0 \end{aligned}\right] \Rightarrow \frac{x}{-40+30} = \frac{y}{-10-0} = \frac{z}{0+20} \text{ or } \frac{x}{1} = \frac{y}{1} = \frac{z}{-2}$$

Eigenvector is $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Eigenvector for $\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e., } \left.\begin{aligned} x + 10y + 5z &= 0 \\ 2x + 5y + 4z &= 0 \end{aligned}\right] \Rightarrow \frac{x}{40-25} = \frac{y}{10-4} = \frac{z}{5-20},$$

or $\quad \frac{x}{5} = \frac{y}{2} = \frac{z}{-5} = k$, Eigenvector $= \begin{bmatrix} 5k \\ 2k \\ -5k \end{bmatrix}$

We get one eigenvector corresponding to repeated root $\lambda_2 = 2 = \lambda_3$.

Eigenvectors corresponding to $\lambda_2 = 2 = \lambda_3$ are not linearly independent.

Similarity transformation is not possible.

Example 79. *Find the eigenvalues and the corresponding eigenvectors for the matrix*

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Solution. $$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda(\lambda - 1)^2 = 0$$

$\lambda = 0, 1, 1$

Matrix equation for eigenvector $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Eigen vector corresponding to eigenvalue $\lambda = 0$.

On putting $\lambda = 0$ in (1), the equation will become

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} x - 6y - 4z = 0 \\ 0x + 4y + 2z = 0 \end{bmatrix}$$

$$\frac{x}{-12+16} = \frac{y}{0-2} = \frac{z}{4-0} \text{ or } \frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$$

Eigenvector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Eigenvector corresponding to eigenvalue $\lambda = 1$.

On putting $\lambda = 1$ in (1), the equation will become

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -6y - 4z = 0 \text{ or } y = -\frac{4}{6}z \quad (z = 3)$$

x can take any value say $x = 1$, $y = -2$, $z = 3$

Eigenvector $X_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

The corresponding eigenvector X_2 and X_3 must be linearly independent.

$$X_3 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

(To make X_3 linearly independent we have choosen $x = 0$) **Ans.**

Exercise 4.13

Non-symmetric matrices with repeated eigenvalues

Find the eigenvalues and eigenvectors of the following matrices :

1. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ **Ans.** $-2, 2, 2;$ $\begin{bmatrix} -4 \\ -1 \\ +7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *(A.M.I.E.T.E., Summer 1997)* **Ans.** $1, 1, 5;$ $k\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ **Ans.** $1, 1, 7;$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **Ans.** $-3, -3, 5;$ $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

5. $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ **Ans.** $-1, -1, 3;$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ *(A.M.I.E.T.E., Summer 1996)* **Ans.** $2, 2, 3;$ $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$ Dependent Eigen vectors

4.31 SYMMETRIC MATRICES WITH NON REPEATED EIGENVALUES

Example 80. *Find the eigenvalues and the corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

Solution. $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

Take $\lambda = -3$, $\quad -27 - 27 + 270 - 216 = 0$

By synthetic division

-3	1	-3	-90	-216
		-3	18	216
	1	-6	-72	0

$$\lambda^2 - 6\lambda - 72 = 0$$

$$(\lambda - 12)(\lambda + 6) = 0$$

$$\lambda = -3, -6, 12$$

Matrix equation for eigenvectors $[A - \lambda I]X = 0$

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

On putting $\lambda = -3$ in eq. (1), it will become

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x + 5y + 4z = 0 \\ 5x + 10y + 5z = 0 \end{cases}$$

$$\frac{x}{25-40} = \frac{y}{20-5} = \frac{z}{10-25} \quad \text{or} \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

Eigenvector $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Eigenvector corresponding to eigenvalue $\lambda = -6$.

Equaton (1) becomes

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{matrix} 4x + 5y + 4z = 0 \\ 5x + 13y + 5z = 0 \end{matrix}$$

$$\frac{x}{25-52} = \frac{y}{20-20} = \frac{z}{52-25} \quad \text{or} \quad \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

Eigenvector $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Eigenvector corresponding to Eigenvalue 12.

Equation (1) becomes

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} -14x + 5y + 4z = 0 \\ 5x - 5y + 5z = 0 \end{cases}$$

$$\frac{x}{25+20} = \frac{y}{20+70} = \frac{z}{70-25} \quad \text{or} \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

Eigenvector $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ **Ans.**

Exercise 4.14

Symmetric matrices with non-repeated eigenvalues

Find the eigen values and eigenvectors of the following matrices:

1. $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ **Ans.** $-2, 4, 6$; $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **Ans.** 2, 3, 6; $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (*A.M.I.E.T.E., Winter 2002*) **Ans.** 0, 3, 15; $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$ **Ans.** $-2, 9, -18$; $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ **Ans.** $-2, 3, 6$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

4.32 SYMMETRIC MATRICES WITH REPEATED EIGENVALUES

Example 81. *Find all the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Soluton. The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or $\quad (2-\lambda)\,[\,(2-\lambda)^2 - 1] + 1\,[-2+\lambda+1] + 1\,[1-2+\lambda] = 0$

or $\quad (2-\lambda)(4-4\lambda+\lambda^2-1) + (\lambda-1) + \lambda - 1 = 0$

or $\quad 8 - 8\lambda + 2\lambda^2 - 2 - 4\lambda + 4\lambda^2 - \lambda^3 + \lambda + 2\lambda - 2 = 0$

or $\quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

or $\quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \qquad ...(1)$

On putting $\lambda = 1$ in (1), the equation (1) is satisfied, so $\lambda - 1$ is one factor of the equation (1). The other factor $(\lambda^2 - 5\lambda + 4)$ is got on dividing (1) by $\lambda - 1$.

or $\quad (\lambda-1)(\lambda^2 - 5\lambda + 4) = 0 \quad$ or $\quad (\lambda-1)(\lambda-1)(\lambda-4) = 0$

$\therefore \quad \lambda = 1, 1, 4$

The eigenvalues are 1, 1, 4.

When $\lambda = 4$ $\quad \begin{pmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

or $\quad \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

On interchanging first and third rows

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_2' \to R_2 + R_1,\ R_3' \to R_3 + 2R_1$ $\qquad R_3' \to R_3 + R_2$

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 - x_2 - 2x_3 = 0$$

$$-3x_2 - 3x_3 = 0$$

Let $x_3 = k$ $\qquad -3x_2 - 3k = 0 \quad \text{or} \quad x_2 = -k$

$$x_1 + k - 2k = 0 \quad \text{or} \quad x_1 = k$$

Let $k = 1$

The eigenvector $\qquad X_1' = [1, -1, 1]'$

When $\lambda = 1$ $\qquad \begin{pmatrix} 2-1 & -1 & 1 \\ -1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$R_2 + R_1$ and $R_3 - R_1$

or $\qquad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \text{or} \quad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$x_1 - x_2 + x_3 = 0$$

Let $x_1 = k_1$ and $x_2 = k_2$

$$k_1 - k_2 + x_3 = 0 \quad \text{or} \quad x_3 = k_2 - k_1$$

Eigenvector is $[k_1, k_2, k_2 - k_1]'$

Choosing $k_1 = 1,\ k_2 = 1$ then eigenvector is $[1, 1, 0]'$, choosing $k_1 = 1,\ k_2 = 2$ then eigen vector $X_2' = [1, 2, -1]'$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 and X_2 since the given matrix is symmetric

$$[1, -1, 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{or} \quad l - m + n = 0 \qquad \ldots (2)$$

$$[1, 1, 0]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{or} \quad l + m + o\,n = 0 \qquad \ldots (3)$$

Solving (2) and (3), we get $\frac{l}{-1} = \frac{m}{1} = \frac{n}{2}$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \textbf{Ans.}$$

Exercise 4.15

Symmetric matrices with repeated eigenvalues

Find the eigenvalues and the corresponding eigen vectors of the following marices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ **Ans.** 0, 0, 14; $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ **Ans.** 1, 3, 3; $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **Ans.** 8, 2, 2; $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

4. Choose the correct or the best of the answers given in the following parts:

(*i*) Two of the eigenvalues of a 3×3 matrix, whose determinant equals 4, are, -1 and $+2$ the third eigen value of the matrix is euqal to

(*a*) -2 (*b*) -1 (*c*) 1 (*d*) 2

(*ii*) If a square matrix A has an eigenvalue λ, then an eigenvalue of the matrix $(kA)^T$ where, $k \neq 0$, is a scalar is

(a) λ / k (b) k / λ (c) $k \lambda$ (d) None of these

(*A.M.I.E.T.E., Summer 1996*)

(*iii*) An eigenvalue of a square matrix A is $\lambda = 0$. Then

(*a*) $|A| \neq 0$; (*b*) A is symmetric ; (*c*) A is singular ;
(*d*) A is skew-symmetric ; (*e*) A is an even order matrix : (*f*) A is an odd order matrix.

(*A.M.I.E.T.E., Summer 1996*)

(*iv*) The matrix A is defined as $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. The eigenvalues of A^2 are

(*a*) $-1, -9, -4,$ (*b*) $1, 9, 4,$ (*c*) $-1, -3, 2,$ (*d*) $1, 3, -2.$

(*v*) If the matrix is $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ then the eigenvalues of $A^3 + 5A + 8I$, are

(*a*) $-1, 27, -8$; (*b*) $-1, 3, -2$; (*c*) $2, 50, -10$; (*d*) $2, 50, 10.$

(*vi*) The matrix A has eigen values $\lambda_i \neq 0$. Then $A^{-1} - 2I + A$ has eigenvalues

(*a*) $1 + 2\lambda_i + \lambda_i^2$ (*b*) $\frac{1}{\lambda_i} - 2 + \lambda_i$ (*c*) $1 - 2\lambda_i + \lambda_i^2$ (*d*) $1 - \frac{2}{\lambda_i} + \frac{1}{\lambda_i^2}$

(*vii*) The eigen values of a matrix A are $1, -2, 3$. The eigenvalues of $3I - 2A + A^2$ are

(*a*) 2, 11, 6 (*b*) 3, 11, 18 (*c*) 2, 3, 6 (*d*) 6, 3, 11.

Ans. (*i*) (*b*), (*ii*) (*c*), (*iii*) (*c*), (*iv*) (*b*), (*v*) (*c*), (*vi*) (*b*), (*vii*) (*a*)

4.33 DIAGONALISATION OF A MATRIX

Theorem. If a square matrix A of order n has n linearly independent eigenvectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Proof. We shall prove the theorem for a matrix of order 3. The proof can be easily extended to matrices of higher order.

Let
$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and X_1, X_2, X_3 the corresponding eigen vectors, where

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

For the eigenvalues λ_1, the eigenvector is given by

$$\left.\begin{aligned} (a_1-\lambda_1)x_1 + b_1y_1 + c_1z_1 &= 0 \\ a_2x_1 + (b_2-\lambda_1)y_1 + c_2z_1 &= 0 \\ a_3x_1 + b_3y_1 + (c_3-\lambda_1)z_1 &= 0 \end{aligned}\right\} \qquad \text{...}(i)$$

∴ We have

$$\left.\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 &= \lambda_1x_1 \\ a_2x_1 + b_2y_1 + c_2z_1 &= \lambda_1y_1 \\ a_3x_1 + b_3y_1 + c_3z_1 &= \lambda_1z_1 \end{aligned}\right\} \qquad \text{...}(ii)$$

Similarly for λ_2 and λ_3 we have

$$\left.\begin{aligned} a_1x_2 + b_1y_2 + c_1z_2 &= \lambda_2x_2 \\ a_2x_2 + b_2y_2 + c_2z_2 &= \lambda_2y_2 \\ a_3x_2 + b_3y_2 + c_3z_2 &= \lambda_2z_2 \end{aligned}\right\} \qquad \text{...}(iii)$$

and

$$\left.\begin{aligned} a_1x_3 + b_1y_3 + c_1z_3 &= \lambda_3x_3 \\ a_2x_3 + b_2y_3 + c_2z_3 &= \lambda_3y_3 \\ a_3x_3 + b_3y_3 + c_3z_3 &= \lambda_3z_3 \end{aligned}\right\} \qquad \text{...}(iv)$$

We consider the matrix
$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

whose columns are the eigenvectors of A.

Then
$$AP = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$= \begin{pmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 & a_1x_3 + b_1y_3 + c_1z_3 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 & a_2x_3 + b_2y_3 + c_2z_3 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 & a_3x_3 + b_3y_3 + c_3z_3 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1x_1 & \lambda_2x_2 & \lambda_3x_3 \\ \lambda_1y_1 & \lambda_2y_2 & \lambda_3y_3 \\ \lambda_1z_1 & \lambda_2z_2 & \lambda_3z_3 \end{pmatrix} \qquad \text{[using results } (ii), (iii) \text{ and } (iv)]$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = PD$$

where D is the Diagonal matrix $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

$\therefore$

$$AP = PD$$

or

$$P^{-1}AP = P^{-1}PD = D$$

Note 1. The square matrix P, which diagonalises A, is found by grouping the eigen vectors of A into square-matrix and the resulting diagonal matrix has the eigen values of A as its diagonal elements.

2. The transformation of a matrix A to $P^{-1}AP$ is known as a *similarity transformation.*

3. The reduction of A to a diagonal matrix is, obviously, a particular case of similarity transformation.

4. The matrix P which diagonalises A is called the modal matrix of A and the resulting diagonal matrix D is known as the spectra matrix of A.

Example 82. *A square matrix A is defined by* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. *Find the modal matrix P and the resulting diagonal matrix D of A.* (A.M.I.E.T.E., *Summer 2001, Summer 1999*)

Solution. The characteristic equation of the matrix A is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

or $\quad (-1-\lambda)(-2\lambda+\lambda^2+1) - 2(-\lambda+1) - 2(-1+2-\lambda) = 0$

or $\quad (-\lambda-1)(\lambda^2-2\lambda+1) + 2\lambda - 2 - 2 + 2\lambda = 0$

or $\quad (-\lambda-1)(\lambda-1)^2 + (4\lambda-4) = 0$

or $\quad (\lambda-1)(\lambda^2-5) = 0, \ \lambda = 1, \ \lambda = \pm\sqrt{5}$

Eigenvalues are $1, -\sqrt{5}, \sqrt{5}$.

Eigenvectors are given by the matrix equation

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 1$, $\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have

$$-2x + 2y - 2z = 0$$
$$x + y + z = 0$$

From first and second equations, we have

$$\frac{x}{2+2} = \frac{y}{-2+2} = \frac{z}{-2-2} \text{ or } \frac{x}{4} = \frac{y}{0} = \frac{z}{-4} = k$$

Hence the eigenvector $X_1 = \begin{bmatrix} 4k \\ 0 \\ -4k \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda = -\sqrt{5}$, $\begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & 0+\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have $(-1+\sqrt{5})x + 2y - 2z = 0,\ x + (2+\sqrt{5})y + z = 0,\ -x - y + \sqrt{5}z = 0$

From first and second equations we have

$$\frac{x}{6+2\sqrt{5}} = \frac{y}{-1-\sqrt{5}} = \frac{z}{1+\sqrt{5}} \text{ or } \frac{x}{1+\sqrt{5}} = \frac{y}{-1} = \frac{z}{1}$$

Hence the eigenvector is $X_2 = \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$.

When $\lambda = \sqrt{5}$, $\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & 0-\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have $(-1-\sqrt{5})x + 2y - 2z = 0,\ x + (2-\sqrt{5})y + z = 0,\ -x - y - \sqrt{5}z = 0$

From first and second equations we have

$$\frac{x}{6-2\sqrt{5}} = \frac{y}{-1+\sqrt{5}} = \frac{z}{1-\sqrt{5}} \text{ or } \frac{x}{1-\sqrt{5}} = \frac{y}{-1} = \frac{z}{1}$$

Hence the eigenvector $X_3 = \begin{bmatrix} 1-\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$.

Modal Matrix $P = \begin{bmatrix} -1 & 1+\sqrt{5} & 1-\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$P^{-1}AP = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ -1 & -2-\sqrt{5} & -1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ -\sqrt{5} & -5-2\sqrt{5} & -\sqrt{5} \\ -\sqrt{5} & 5-2\sqrt{5} & -\sqrt{5} \end{bmatrix} \begin{bmatrix} -1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix} = D.$$

Example 83. *Let* $A = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

Find a similarity transformation that diagonalises matrix A. *(Allahabad 1997)*

Solution. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristic Eqn. is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -(2+\lambda) & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(2+\lambda)[(1-\lambda)(-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + (1-\lambda)] = 0$$

$$\Rightarrow -(2+\lambda)[-\lambda + \lambda^2 - 12] + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$$

$$\Rightarrow 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0 \Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda^2 + 6\lambda + 9) = 0, \Rightarrow (\lambda - 5)(\lambda + 3)^2 = 0$$

$$\Rightarrow \lambda = 5, -3, -3.$$

Eigenvector for $\lambda = 5$ is

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ \\ R_3 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_3 + 2R_2$$

$$-x - 2y - 5z = 0$$

$$-8y - 16z = 0$$

Let $z = 1 \therefore$ $\quad -8y - 16 = 0$ or $y = -2$

$$-x + 4 - 5 = 0 \text{ or } x = -1$$

Eigenvector is $(-1, -2, 1)'$

For $\lambda = -3$ $\quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$x + 2y - 3z = 0$$

Let $\quad z = 0, y = -1, x - 2 + 0 = 0$ or $x = 2$

Eigenvector is $[2 \quad -1 \quad 0]'$

Let $\quad z = 1, y = 0, x + 0 - 3 = 0$ or $x = 3$

Eigenvector is $(3, \quad 0, \quad 1)'$

We have got three eigen vectors *i.e.*,

$$(-1, -2, 1)', (2, -1, 0)', (3, 0, 1)'$$

Three independent eigen vectors, writing as three column vectors, we get the required transformation matrix.

$$B = \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B^{-1} = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ 2 & -4 & -6 \\ 1 & 2 & 5 \end{bmatrix}$$

$$B^{-1}AB = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ 2 & -4 & -6 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 6 \end{bmatrix}\begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ -6 & 12 & 18 \\ -3 & -6 & -15 \end{bmatrix}\begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 40 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$ **Ans.**

Example 84. *Let* $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Find matrix P such that $P^{-1}AP$ is diagonal matrix.

Solution. The characteristic equation of the matrix A is

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

or $(6-\lambda)[9+\lambda^2-6\lambda-1]+2[-6+2\lambda+2]+2[2-6+2\lambda] = 0$

or $(6-\lambda)(\lambda^2-6\lambda+8)-8+4\lambda-8+4\lambda = 0$

or $6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda-16+8\lambda = 0$

or $-\lambda^3+12\lambda^2-36\lambda+32 = 0$ or $\lambda^3-12\lambda^2+36\lambda-32 = 0$

or $(\lambda-2)^2(\lambda-8) = 0$ or $\lambda = 2, 2, 8$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \frac{1}{2}R_1$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad 2x_1 - x_2 + x_3 = 0$$

This equation is satisfied by $x_1 = 0, x_2 = 1, x_3 = 1$ and $x_1 = 1, x_2 = 3, x_3 = 1$.

Eigen vectors are $[0, 1, 1]', [1, 3, 1]'$.

Eigenvector for $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; \frac{1}{2}R_1$$

$R_2 + 2R_1, R_3 - 2R_1$

or $$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 - x_3 = 0 \text{ and } x_2 + x_3 = 0$$

Let $x_3 = 1, x_2 = -1$ and so $x_1 = 2$.

Eigenvector is $[2, -1, 1]'$

$\therefore$ $$P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \; P^{-1} = -\frac{1}{6}\begin{bmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

Now $$P^{-1}AP = \frac{1}{6}\begin{bmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$ **Ans.**

Example 85. *The matrix* $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ *is transformed to the diagonal form* $D = T^{-1}AT$, *where* $T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. *Find the value of* θ *which gives this diagonal transformation.*

Solution. $T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \therefore T^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Now $$T^{-1}AT = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos\theta - h\sin\theta & h\cos\theta - b\sin\theta \\ a\sin\theta + h\cos\theta & h\sin\theta + b\cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta - 2h\sin\theta\cos\theta + b\sin^2\theta & (a-b)\sin\theta\cos\theta - h\sin^2\theta + h\cos^2\theta \\ (a-b)\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta & a\sin^2\theta + 2h\sin\theta\cos\theta + b\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta - h\sin 2\theta + b\sin^2\theta & (a-b)\sin\theta\cos\theta + h\cos 2\theta \\ (a-b)\sin\theta\cos\theta + h\cos 2\theta & a\sin^2\theta + h\sin 2\theta + b\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \quad \text{Being diagonal matrix}$$

$\therefore$ $(a-b)\sin\theta\cos\theta + h\cos 2\theta = 0$

or $\quad \dfrac{a-b}{2}\sin 2\theta + h\cos 2\theta = 0 \quad$ or $\quad \dfrac{a-b}{2}\sin 2\theta = -h\cos 2\theta$

or $\quad \tan 2\theta = \dfrac{2h}{b-a} \quad$ or $\quad \theta = \dfrac{1}{2}\tan^{-1}\dfrac{2h}{b-a}$ **Ans.**

Exercise 4.16

1. Find the matrix B which transforms the matrix

$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ to a diagonal matrix. *(A.M.I.E.T.E., winter 2000)* **Ans.** $B = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

2. For the matrix $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, determine a matrix P such that $P^{-1}AP$ is diagonal matrix.

(A.M.I.E.T.E., Summer 1997) $\quad P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$

3. Determine the eigen values and the corresponding egienvectors of the matrix

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

Hence find the matrix B such that $B^{-1}AB$ is diagonal matrix. *(A.M.I.E.T.E., Winter 1997)*

Ans. $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

4. Diagonalise the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5. Reduce the following matrix A into a diagonal matrix

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ *(A.M.I.E.T.E., Winter 2002, Summer 2000)* **Ans.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

4.34 POWERS OF A MATRIX

We can obtain powers of a matrix by using diagonalisation.

We know that $\quad D = P^{-1}AP$

Where A is the square matrix and P is a non-singular matrix.

$$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P$$

Similarly $\quad D^3 = P^{-1}A^3P$

In general $\quad D^n = P^{-1}A^nP \quad \ldots (1)$

Pre-multiply (1) by P and post-multiply by P^{-1}

$$\begin{aligned} PD^nP^{-1} &= P(P^{-1}A^nP)P^{-1} \\ &= (PP^{-1})A^n(PP^{-1}) \\ &= A^n \end{aligned}$$

Procedure: (1) Find eigenvalues for a square matrix A.

(2) Find eigenvectors to get the modal matrix P.
(3) Find the diagonal matrix D, by the formula D.
(4) Obtain A^n by the formula $A^n = P D^n P^{-1}$.

Example 86. *Find a matrix P which transform the matrix* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ *to diagonal form. Hence* A^4.

Solution. Characteristic equation of the matrix A is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \text{or} \quad (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

For $\lambda = 1$, eigenvectors are given by

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 0x_1 + 0x_2 - x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}\right] \Rightarrow \frac{x_1}{0+1} = \frac{x_2}{-1+0} = \frac{x_3}{0} \text{ or } x_1 = 1,\ x_2 = -1,\ x_3 = 0$$

Eigenvector is [1, –1, 0].

For $\lambda = 2$, eigenvectors are given by

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} x_1 + 0x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + x_3 = 0 \end{array}\right] \Rightarrow \frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0} \text{ or } x_1 = -2,\ x_2 = 1,\ x_3 = 2$$

Eigenvector is [–2, 1, 2].

For $\lambda = 3$, eigenvectors are given by

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} -2x_1 + 0x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{array}\right] \Rightarrow \frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0} \text{ or } x_1 = -1,\ x_2 = 1,\ x_3 = 2$$

Eigenvector is [–1, 1, 2].

Modal matrix $P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ and $P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

Now $$P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix} \quad \textbf{Ans.}$$

Exercise 4.17

Find a matrix P which transform the following matrices to diagonal form. Hence calculate the power matrix.

1. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, calculate A^4. **Ans.** $\begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}$

2. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, calculate A^4. **Ans.** $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$

3. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, calculate A^6. **Ans.** $\begin{bmatrix} 1446 & -1365 & 1365 \\ -1365 & 1446 & -1365 \\ 1365 & -1365 & 1446 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$, calculate A^8. **Ans.** $\begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$

4.35 SYLVESTER'S THEOREM

Let $P(A) = C_0 A^n + C_1 A^{n-1} + C_2 A^{n-2} + \dots + C_{n-1} A + C_n I$

and $|\lambda I - A| = f(\lambda)$ and Adjoint matrix of $[\lambda I - A] = [f(\lambda)]$

$$Z(\lambda) = \frac{[f(\lambda)]}{f'(\lambda)} = \frac{\text{Adjoint matrix of } [\lambda I - A]}{f'(\lambda)}$$

Then according to Sylvester's theorem

$$P(A) = P(\lambda_1).\,Z(\lambda_1) + P(\lambda_2).\,Z(\lambda_2) + P(\lambda_3).\,Z(\lambda_3) + \dots$$

$$= \sum_{r=1}^{n} P(\lambda_r).\,Z(\lambda_r)$$

Example 87. *If* $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, *find* A^{100}.

Solution $f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 1 \end{vmatrix}$

or $f(\lambda) = (\lambda - 2)(\lambda - 1) = 0$ or $\lambda_1 = 1,\ \lambda_2 = 2$

$f(\lambda) = \lambda^2 - 3\lambda + 2,\ f'(\lambda) = 2\lambda - 3$ or $\lambda_1 = 1,\ \lambda_2 = 2$

$f'(2) = 4 - 3 = 1, \quad f'(1) = 2 - 3 = -1$

$[f(\lambda)]$ = Adjoint of the matrix of the matrix $[\lambda I - A]$

$$Z(\lambda_1) = Z(1) = \frac{[f(1)]}{f'(1)} = \frac{1}{-1}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Z(\lambda_2) = Z(2) = \frac{[f(2)]}{f'(2)} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

By Sylvester theorem $P(A) = P(\lambda_1).\,Z(\lambda_1) + P(\lambda_2).\,Z(\lambda_2)$

$$A^{100} = P(\lambda_1)\,Z(\lambda_1) + P(\lambda_2)\,Z(\lambda_2)$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

5. Reduce to sum of squares

$$Q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_2x_3$$

Ans. $Q = y_1^2 - 2y_2^2 + 9y_3^2, \quad P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

6. Express the following quadratic form as "sum of squares" by congruent transformation and write down the corresponding linear transformation

$$Q = 10x_1^2 + x_2^2 + x_3^2 - 6x_1x_2 - 2x_2x_3 + x_3x_1$$

Ans. $10y_1^2 + \frac{1}{10}y_2^2, \quad x_1 = y_1 + \frac{3}{10}y_2, \quad x_2 = y_2 + y_3, \quad x_3 = y_3$

7. Reduce to the diagonal matrix by rational congruent transformation and interpret the result in terms of quadratic form.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

Ans. $Q = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 6x_2x_3 - 2x_3x_1$

$Q = y_1^2 - 4y_2^2 + \frac{25}{4}y_3^2$

4.38 DIFFERENTIATION AND INTEGRATION OF MATRICES

If the elements of a matrix A are function of scalar variable t, the matrix is called a matrix function of t.

$$A = A(t) = [a_{ij}(t)]$$

The differential coefficient of A w.r.t. "t" is defined as

$$\frac{d}{dt}(A) = \left[\frac{d}{dt}(a_{ij})\right]$$

Hence the elements of the differentiated matrix $\frac{dA}{dt}$ are the derivatives of the corresponding elements of A.

$$\frac{d}{dt}A(t) = \begin{bmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} & \cdots & \frac{da_{1n}}{dt} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} & \cdots & \frac{da_{2n}}{dt} \\ \frac{da_{n1}}{dt} & \frac{da_{n2}}{dt} & \cdots & \frac{da_{nn}}{dt} \end{bmatrix}$$

It is easy to prove that

$$\frac{d}{dt}(A)(B) = A.\frac{dB}{dt} + B\frac{dA}{dt}$$

The integral of the matrix A is defined as

$$\int A\,dt = \left[\int (a_{ij})\,dt\right]$$

Thus the integral of A is obtained by integrating each element of A.

Power series. Let A be a square matrix with all eigenvalues less than 1 in absolute value, then $a_0 I + a_1 A + a_2 A^2 + \ldots\ldots$ is convergent.

The following series are also convergent

$$e^A = I + \frac{A}{\lfloor 1} + \frac{1}{\lfloor 2} A^2 + \ldots\ldots$$

$$\cos A = 1 - \frac{1}{\lfloor 2} A^2 + \frac{1}{\lfloor 4} A^4 - \ldots\ldots$$

$$\sin A = A - \frac{1}{\lfloor 3} A^3 + \frac{1}{\lfloor 5} A^5 - \ldots\ldots$$

$$(1 - A)^{-1} = 1 + A + A^2 + \ldots\ldots$$

Example 93. *Prove that* $\frac{d}{dt}(e^{tA}) = A\, e^{tA}$ *if* $e^{tA} = I + \frac{tA}{\lfloor 1} + \frac{1}{\lfloor 2}(tA)^2 + \frac{1}{\lfloor 3}(tA)^3 + \ldots\ldots$

Solution.
$$\frac{d}{dt}(e^{tA}) = \frac{d}{dt}\left[1 + \frac{tA}{\lfloor 1} + \frac{1}{\lfloor 2} t^2 A^2 + \frac{1}{\lfloor 3} t^3 A^3 + \ldots\ldots\right]$$

$$= \frac{d}{dt}(1) + \frac{d}{dt}(tA) + \frac{d}{dt}\left(\frac{1}{\lfloor 2} t^2 A^2\right) + \frac{d}{dt}\left(\frac{1}{\lfloor 3} t^3 A^3\right) + \ldots\ldots$$

$$= 0 + A + \frac{1}{\lfloor 1} t A^2 + \frac{1}{\lfloor 2} t^2 A^3 + \ldots\ldots$$

$$= A\left[1 + \frac{1}{\lfloor 1} tA + \frac{1}{\lfloor 2} t^2 A^2 + \ldots\ldots\right] = A\, e^{tA}$$ **Ans.**

Example 94. *Solve* $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 12x = 0.$ ... (1)

$x(0) = 0;\ x'(0) = 8$ *by Matrix Method.*

Solution. Let $x = x_1$ and $\frac{dx_1}{dt} = x_2$ ... (2)

(1) becomes

$$\therefore \quad \frac{d}{dt}\left(\frac{dx_1}{dt}\right) = -4\frac{dx_1}{dt} + 12x_1$$ ... (3)

or $$\frac{dx_2}{dt} = 12x_1 - 4x_2$$

(2) and (3) are written in matrix form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 12 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From R.H.S. we have to find eigenvector.

Characteristic equation is $\begin{bmatrix} 0-\lambda & 1 \\ 12 & -4-\lambda \end{bmatrix} = 0$

or $-\lambda(-4-\lambda) - 12 = 0$ or $\lambda^2 + 4\lambda - 12 = 0$

or $(\lambda - 2, \ldots + 6) = 0 \Rightarrow \lambda = 2, -6$

For $\lambda = 2$ and $\lambda = -6$, eigenvectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -6 \end{bmatrix}$

Matrix of eigenvectors $= P = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}$, $P^{-1} = \frac{1}{8}\begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix}$

Now $\quad P e^{\lambda t} P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-6t} \end{bmatrix}\frac{1}{8}\begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix}$

$$= \frac{1}{8}\begin{bmatrix} e^{2t} & e^{-6t} \\ 2e^{2t} & -6e^{-6t} \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 6e^{2t}+2e^{-6t} & e^{2t}-e^{-6t} \\ 12e^{2t}-12e^{-6t} & 2e^{2t}+6e^{-6t} \end{bmatrix}$$

By initial conditions $\qquad x(0) = 0,\ x'(0) = 8$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 6e^{2t}+2e^{-6t} & e^{2t}-e^{-6t} \\ 12e^{2t}-12e^{-6t} & 2e^{2t}+6e^{-6t} \end{bmatrix}\begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} e^{2t}-e^{-6t} \\ 2e^{2t}+6e^{-6t} \end{bmatrix}$$

$x_1 = x = e^{2t} - e^{-6t}$, $x_2 = \frac{dx}{dt} = 2e^{2t} + 6e^{-6t}$ **Ans.**

Example 95. *Solve by matrix method*

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0,\ x(0) = 1, x'(0) = 2.$$

Solution. $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$ ... (1)

Let $x = x_1$ and $\frac{dx_1}{dt} = x_2$ ... (2)

On substitution (1) becomes

$$\frac{dx_2}{dt} = 5x_2 - 6x_1 \text{ or } \frac{dx_2}{dt} = -6x_1 + 5x_2 \quad \text{... (3)}$$

Equations (2) and (3) are written in a single matrix equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From R.H.S. we have to find eigenvector.

Characteristic equation is $\begin{vmatrix} 0-\lambda & 1 \\ -6 & 5-\lambda \end{vmatrix} = 0$

$$-\lambda(5-\lambda) + 6 = 0 \text{ or } \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2,\ 3$$

Eigenvectors for $\lambda = 2$ and $\lambda = 3$ are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Matrix of eigenvectors $= P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$. $P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$$P e^{\lambda t} P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3e^{2t}-2e^{3t} & -e^{2t}+e^{3t} \\ 6e^{2t}-6e^{3t} & -2e^{2t}+3e^{3t} \end{bmatrix}$$

By initial conditions $x(0) = 1$ and $x'(0) = 2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3e^{2t}-2e^{3t} & -e^{2t}+e^{3t} \\ 6e^{2t}-6e^{3t} & -2e^{2t}+3e^{3t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{2t} - 2e^{3t} & -2e^{2t} + 2e^{3t} \\ 6e^{2t} - 6e^{3t} & -4e^{2t} + 6e^{3t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$x_1 = x = e^{2t}$$

$$x_2 = \frac{dx}{dt} = 2e^{2t} \qquad \textbf{Ans.}$$

Example 96. *Use matrices to solve the differential equation*

$$\frac{d^2 y}{dx^2} + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution. Let

$$y = y_1, \quad \frac{dy_1}{dx} = y_2 \qquad \dots (1)$$

$$\frac{d^2 y}{dx^2} + 4y = 0 \text{ or } \frac{d}{dx}\left(\frac{dy_1}{dx}\right) + 4y_1 = 0 \text{ or } \frac{dy_2}{dx} = -4y_1 \qquad \dots (2)$$

Differential equations (1) and (2) are written in matrix form

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The characteristic equation of $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ is

$$\begin{bmatrix} 0-\lambda & 1 \\ -4 & 0-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm i2$$

Eigenvector for $\lambda = i2$

$$\begin{bmatrix} 0-i2 & 1 \\ -4 & 0-i2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -i2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-i2\,x_1 + x_2 = 0 \text{ or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i2 \end{bmatrix}$$

Eigenvector for $\lambda = -i2$

$$\begin{bmatrix} 0+i2 & 1 \\ -4 & 0+i2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} i2 & 1 \\ -4 & i2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } i2\,x_1 + x_2 = 0 \text{ or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i2 \end{bmatrix}$$

Let $P = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} - P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4i} \\ \frac{1}{2} & -\frac{1}{4i} \end{bmatrix}$

$$P e^{\lambda x} P^{-1} = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}\begin{bmatrix} e^{i2x} & 0 \\ 0 & e^{-i2x} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{4i} \\ \frac{1}{2} & -\frac{1}{4i} \end{bmatrix} = \begin{bmatrix} e^{i2x} & e^{-i2x} \\ 2ie^{i2x} & -2ie^{-2x} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{4i} \\ \frac{1}{2} & -\frac{1}{4i} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{i2x} + \frac{1}{2}e^{-i2x} & \frac{1}{4i}e^{i2x} - \frac{1}{4i}e^{-i2x} \\ ie^{i2x} - ie^{-2ix} & \frac{1}{2}e^{i2x} + \frac{1}{2}e^{-i2x} \end{bmatrix} = \begin{bmatrix} \cos 2x & \frac{1}{2}\sin sx \\ -2\sin 2x & \cos 2x \end{bmatrix}$$

Applying the initial conditions, we get

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos 2x & \frac{1}{2}\sin 2x \\ -2\sin 2x & \cos 2x \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 2x \\ -2\sin 2x \end{bmatrix}$$ or $y_1 = \cos 2x$ and $y_2 = -2\sin 2x$ **Ans.**

EXERCISE 4.20

Solve the following differential equations by matrix method:

1. $\frac{d^2y}{dx^2} - \frac{4\,dy}{dx} + 3y = 0,\ y(0) = 2,\ y'(0) = 1$ **Ans.** $y = \frac{5}{2}e^x - \frac{e^{3x}}{2}$

2. $\frac{d^2y}{dx^2} - \frac{3\,dy}{dx} + 2y = 0,\ y(0) = 5,\ y'(0) = 8$ **Ans.** $y = 2e^x + 3e^{2x}$

3. $\frac{d^2y}{dx^2} + \frac{5\,dy}{dx} - 14y = 0,\ y(0) = 2,\ y'(0) = -5$ **Ans.** $y = e^{2x} + e^{-7x}$

4. $\frac{d^2y}{dx^2} + \mu^2 y = 0,\ y(0) = 1,\ y'(0) = \mu$ **Ans.** $y = \cos x + \sin x$

5. $\frac{d^2y}{dx^2} + 9y = 0,\ y(0) = 1,\ y'(0) = 3$ **Ans.** $y = \cos x + \sin x$